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2016 Laser Phys. Lett. 13 116202

(http://iopscience.iop.org/1612-202X/13/11/116202)

IOP Publishing | Astro Ltd Laser Physics Letters
Laser Phys. Lett. 13 (2016) 116202 (6pp) doi:10.1088/1612-2011/13/11/116202

Letter

Controllably accelerating and decelerating Airy–Bessel–Gaussian wave packets

Fu Deng[1,2], Weihao Yu[1,2] and Dongmei Deng[1,2]

[1] Guangdong Provincial Key Laboratory of Nanophotonic Functional Materials and Devices, South China Normal University, Guangzhou 510631, People's Republic of China
[2] CAS Key Laboratory of Geospace Environment, University of Science & Technology of China, Chinese Academy of Sciences, Hefei 230026, People's Republic of China

E-mail: dmdeng@263.net

Received 22 May 2016, revised 25 August 2016
Accepted for publication 25 September 2016
Published 29 September 2016

Abstract
By solving the (3 + 1)D free-space Schrödinger equation in polar coordinates analytically, we have investigated the propagation of 3D controllably accelerating and decelerating Airy–Bessel–Gaussian (CAiBG) wave packets, even CAiBG wave packets, odd CAiBG wave packets and the superposition of several CAiBG wave packets in free space. The CAiBG wave packets are constructed with the Airy pulses with initial velocity in temporal domain and the Bessel–Gaussian beams in space domain. Due to the initial velocity on Airy pulses, we can obtain decelerating and accelerating Airy–Bessel–Gaussian wave packets by selecting different initial velocities. Moreover, by superposing several CAiBG wave packets, we can obtain the rotating wave packets.

Keywords: spatiotemporal Airy–Bessel–Gaussian wave packets, accelerating and decelerating, initial velocity, linear (3 + 1)D Schrödinger equation

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the finite energy Airy beams were first demonstrated theoretically and produced experimentally by Siviloglou and Christodoulides in 2007 [1, 2], they have attracted considerable interest in optics along the world [3–11]. As a solution of the Schrödinger equation without external potential [12], Airy beams have unusual propagation characteristics including self-acceleration [2], self-healing [3] and quasi-diffraction-free [4], which lead to their many potential applications in optics such as micro-manipulation [13–15], filamentation [16], Airy plasmon generation [17], and vacuum electron acceleration [18, 19]. Recently, some studies of Airy beams with additional initial conditions including initial frequency chirp [20], initial velocity [21] and initial quadratic phase modulation [22] have been investigated.

On the other hand, the generation of Airy beams in light bullets that are impervious to both dispersion and diffraction has gained increasing attention [23–28]. Chong *et al* first experimentally realized the 3D Airy–Bessel wave packets by combining spatial Bessel beams and temporal Airy pulses in 2010 [23]. Then in the same year, Abdollahpour *et al* demonstrated spatiotemporal Airy light bullets with an Airy waveform both in temporal and space domains in the linear and nonlinear regimes [24]. Eichelkrant *et al* investigated the oblique spatiotemporal Airy wave packets with oblique Airy wave-form both in temporal and space domains and oblique spatiotemporal Airy–Bessel wave packets with oblique Airy pulses in temporal and oblique Airy beams in space domain using polar rotations in bidispersive optical media [25]. The other spatiotemporal wave packets with Airy pulses including Airy-vortex wave packets [26], Airy–Laguerre–Gaussian wave packets [27], Airy–Hermite–Gaussian wave packets [28], have been demonstrated. But almost all researches of these spatiotemporal wave packets using the Airy pulses without initial velocity. Therefore, it will be interesting to investigate the spatiotemporal wave packets which are generated by combing the Airy pulses with initial velocity in temporal domain and other beams in space domain.

In this letter, we construct the Airy pulses with initial velocity in temporal domain and the Bessel–Gaussian vortex beams in space domain to obtain the controllably accelerating and decelerating Airy–Bessel–Gaussian (CAiBG) wave packets, and investigate their propagation properties in free space. In addition, we also discuss the propagation of the superposition of several CAiBG wave packets.

2. The model of CAiBG wave packets

In the 3D diffractive and dispersive optical paraxial system, the propagation of the 3D light wave packets along the direction z in free space is obedient to the (3+1)D linear Schrödinger equation, which can be read in normalized form in polor coordinates [2, 23, 24]:

$$\mathrm{i}\frac{\partial U}{\partial Z}+\frac{1}{2}\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\frac{\partial U}{\partial R}+\frac{1}{R^2}\frac{\partial^2 U}{\partial \varphi^2}+\frac{\partial^2 U}{\partial t^2}\right)=0, \quad (1)$$

where $U(Z,R,\varphi,t)$ denotes the light field amplitude, $R=\sqrt{x^2+y^2}/w_0$ and t represent the normalized transverse coordinates, respectively. $Z=z/kw_0^2$ is the normalized propagation distance with the Rayleigh length kw_0^2, where $k=2\pi n_0/\lambda_0$ is the wave number (λ_0 is the vacuum wavelength and n_0 is the refractive index of free space) and w_0 is the beam-waist width of a Gaussian beam in the $x-y$ plane, φ is the azimuth angle. Some parameters are chosen as $w_0=100$ μm, $n_0=1$ and $\lambda_0=632.8$ nm.

To obtain one solution of equation (1), we resort to a separation of variables and assume a solution of the form:

$$U=M(R,\varphi,Z)\Phi_G(R,Z)P(t,Z)\mathrm{e}^{-\mathrm{i}\beta Z}, \quad (2)$$

where $\Phi_G(R,Z)=\frac{-\mathrm{i}}{Z-i}\mathrm{e}^{\frac{\mathrm{i}R^2}{2(Z-i)}}$, β is the propagation constant. We can obtain the following two differential equations by substituting equation (2) into equation (1):

$$\frac{\mathrm{i}}{P}\frac{\partial P}{\partial Z}+\frac{1}{2P}\frac{\partial^2 P}{\partial T^2}=0, \quad (3)$$

$$\frac{\mathrm{i}}{M}\frac{\partial M}{\partial Z}+\beta+\frac{1}{2M}\frac{\partial^2 M}{\partial R^2}+\frac{\mathrm{i}R}{Z-i}\frac{1}{M}\frac{\partial M}{\partial R}+\frac{1}{2RM}\frac{\partial M}{\partial R}+\frac{1}{2R^2M}\frac{\partial^2 M}{\partial \varphi^2}=0. \quad (4)$$

Firstly, we concentrate on equation (3). It is obvious that equation (3) is the one dimensional Schrödinger equation without any potential. The accelerating Airy function as a solution of equation (3) has been studied extensively [1–4]. Here, we investigate the dynamics of finite-energy Airy pulses with initial velocity in the temporal t domain, which can be written as $P(0,t)=Ai(t)\mathrm{e}^{at}\mathrm{e}^{\mathrm{i}v_0t}$ in the input plane, where v_0 represents the magnitude of the velocity vector [21], $Ai(\cdot)$ is the Airy function and a is the truncation of the Airy function [1, 2].

We can obtain the solution by directly solving equation (3) with such initial condition as:

$$P(Z,t)=Ai\left(t+\mathrm{i}aZ-v_0Z-\frac{Z^2}{4}\right)\mathrm{e}^{a\left(t-v_0Z-\frac{Z^2}{2}\right)}\times\mathrm{e}^{\left[\frac{\mathrm{i}}{2}(a^2Z-v_0^2Z+2v_0t-v_0Z^2+tZ)-\frac{\mathrm{i}Z^3}{12}\right]}. \quad (5)$$

From equation (5), we can find the trajectory of such Airy pulses during propagation, in the form: $t=v_0Z+Z^2/4$. Having found the analytical expression of the propagation of finite-energy Airy pulses with initial velocity in the temporal t domain in equation (5), figure 1 shows that the temporal evolution of Airy pulses as a function of propagation distance for different values of initial velocity. When $v_0=0$, figure 1(b) is consistent with the expression for spatial Airy beams [1, 2], which can also be seen in equation (5). For the case of $v_0=-3$ is shown in figure 1(a), the trajectory of such Airy pulses decelerate in propagation. While for the case of $v_0=3$ shown in figure 1(c), the Airy pulses are distorted quickly and accelerate faster. It is also interesting to find the Airy pulses at first display deceleration and then display acceleration when $-3<v_0<0$, which is predicted in [21].

Next, we consider the solution of equation (4). To find a solution of equation (4), here we assume $R_1=\frac{-2\mathrm{i}}{Z-i}hR$ (h is an arbitrary stretching coefficient), $M=M_1(Z)M_2(R_1)\phi(\varphi)$, and obtain the following two Schrödinger equations with lower dimensions:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\varphi^2}+m^2\phi=0, \quad (6)$$

$$\frac{\mathrm{i}}{M_1}\frac{\mathrm{d}M_1}{\mathrm{d}Z}+\beta-\frac{2h^2}{(Z-i)^2}\frac{1}{M_2}\frac{\mathrm{d}^2M_2}{\mathrm{d}R_1^2}-\frac{2h^2}{(Z-i)^2}\frac{1}{M_2R_1}\frac{\mathrm{d}M_2}{\mathrm{d}R_1}+\frac{-2h^2m^2}{(Z-i)^2R_1^2}=0, \quad (7)$$

where m is the topological charge of the vortex. We can obtain the solution of equation (6) by solving it directly: $\phi(\varphi)=\mathrm{e}^{\pm \mathrm{i}m\varphi}$, where the signs $\pm$ denote the positive and negative vortices.

Finally, we search for the solution of equation (7). By multiplying $\frac{(Z-i)^2}{2h^2}$ on both sides of equation (7) and after some algebras, we can obtain

$$\frac{(Z-i)^2}{2h^2}\frac{\mathrm{i}}{M_1}\frac{\mathrm{d}M_1}{\mathrm{d}Z}+\frac{(Z-i)^2}{2h^2}\beta=-\beta_1, \quad (8)$$

$$-\frac{1}{M_2}\frac{\mathrm{d}^2M_2}{\mathrm{d}R_1^2}-\frac{1}{M_2R_1}\frac{\mathrm{d}M_2}{\mathrm{d}R_1}+\frac{m^2}{R_1^2}=\beta_1. \quad (9)$$

Equation (8) can be solved directly as $M_1=c_0\mathrm{e}^{-\frac{2\mathrm{i}h^2\beta_1}{Z-i}+\mathrm{i}\beta Z}$, where c_0 is an arbitrary constant. And equation (9) can be written as:

$$R_1^2\frac{\mathrm{d}^2M_2}{\mathrm{d}R_1^2}+R_1\frac{\mathrm{d}M_2}{\mathrm{d}R_1}+(\beta_1R_1^2-m^2)M_2=0. \quad (10)$$

We let $\rho=\sqrt{\beta_1}R_1$ and obtain

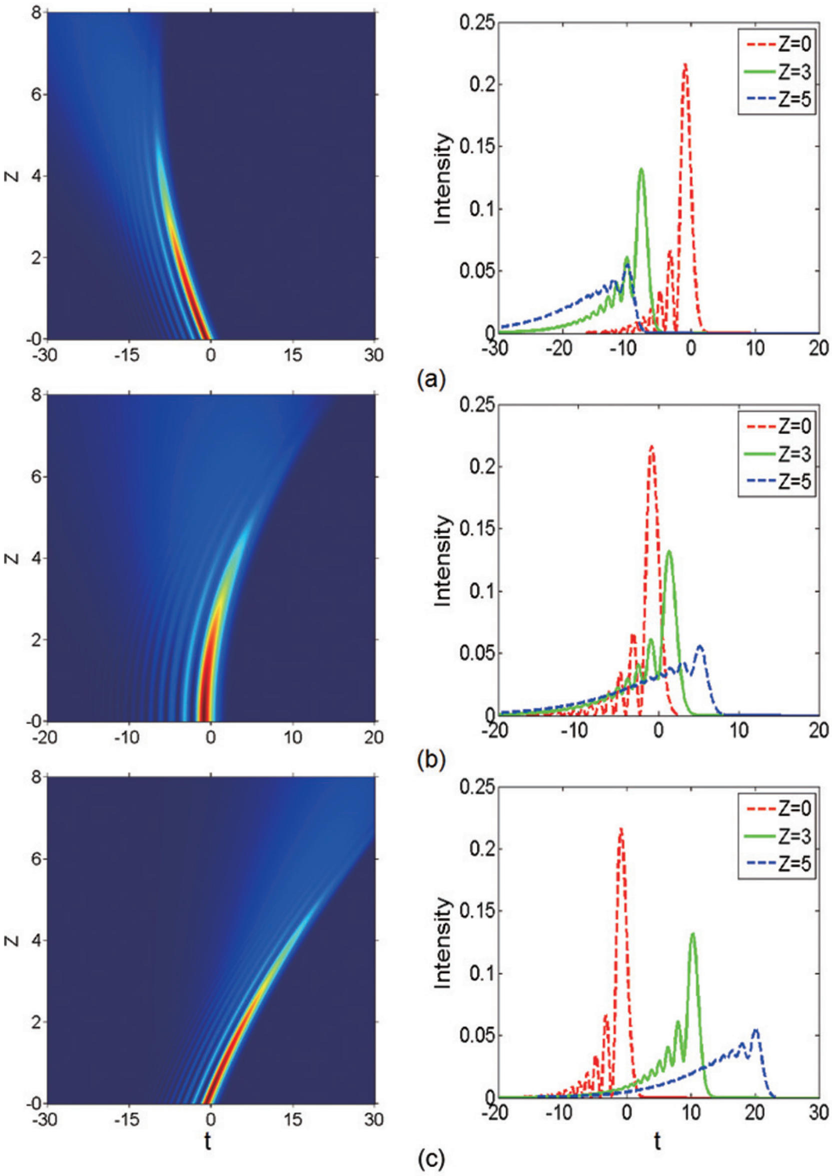

Figure 1. The temporal evolution of Airy pulses as a function of propagation distance for different values of initial velocities. The left column denotes the contour-plot intensity of the finite-energy Airy beams with different initial velocities during propagation, and the right column denotes the intensity profile at various distances with different initial velocities. (a)–(c) Represent $v_0 = -3$, $v_0 = 0$ and $v_0 = 3$, respectively. The decay constant is chosen as $a = 0.15$.

$$\rho^2 \frac{\mathrm{d}^2 M_2}{\mathrm{d}\rho^2} + \rho \frac{\mathrm{d}M_2}{\mathrm{d}\rho} + (\rho^2 - m^2) M_2 = 0. \quad (11)$$

It is obvious that equation (11) is the well-known m-order Bessel function of the first kind [29], whose solution can be obtained as $M_2 = J_m(\rho)$. Combining M_1, M_2 and ϕ, we obtain the solution of equation (4):

$$M = c_0 \mathrm{e}^{-\frac{2ih^2\beta_1}{Z-i}} J_m\left(\sqrt{\beta_1}\,\frac{-2\mathrm{i}}{Z-i} hR\right) \mathrm{e}^{\pm \mathrm{i}m\varphi} \times \frac{-\mathrm{i}}{Z-i} \mathrm{e}^{\frac{\mathrm{i}R^2}{2(Z-i)}}. \quad (12)$$

Finally, by substituting equations (5) and (12) into equation (1), the exact solution of equation (2) can be written as

$$U = c_0 \mathrm{e}^{-\frac{2ih^2\beta_1}{Z-i}} J_m\left(\sqrt{\beta_1}\,\frac{-2\mathrm{i}}{Z-i} hR\right) \frac{-\mathrm{i}}{Z-i} \mathrm{e}^{\frac{\mathrm{i}R^2}{2(Z-i)}} \times Ai\left(t + \mathrm{i}aZ - v_0 Z - \frac{Z^2}{4}\right) \mathrm{e}^{\left[a\left(t - v_0 Z - \frac{Z^2}{2}\right) - \frac{\mathrm{i}Z^3}{12}\right]} \times \mathrm{e}^{\frac{\mathrm{i}}{2}(a^2 Z - v_0^2 Z + 2v_0 t - v_0 Z^2 + tZ)} \mathrm{e}^{\pm \mathrm{i}m\varphi}. \quad (13)$$

3. Controllable Airy–Bessel–Gaussian wave packets

3.1. Vortex CAiBG wave packets

Figure 2 displays the wave packet intensity ($|U|^2$) of the CAiBG wave packets with $h = 1$, $c_0 = 1$, and $m = 2$ in free space.

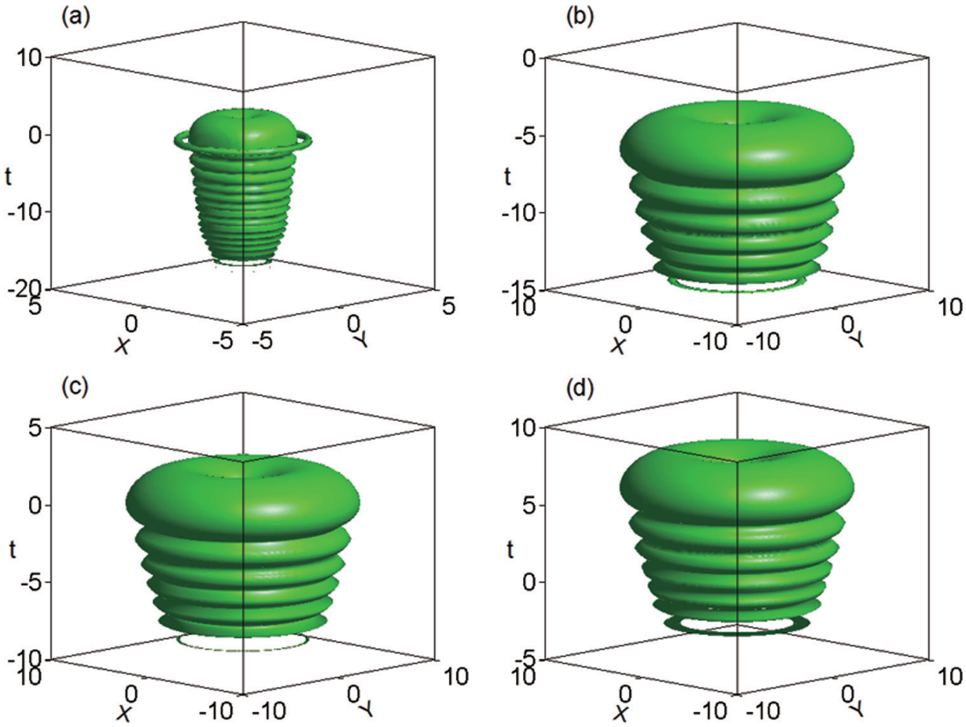

Figure 2. Snapshots describing the evolution of the spatiotemporal CAiBG wave packets with $m = 2$, taken (a) at $Z = 0$, (b)–(d) at $Z = 2$. (b)–(d) Represent $v_0 = -3$, $v_0 = 0$ and $v_0 = 3$, respectively.

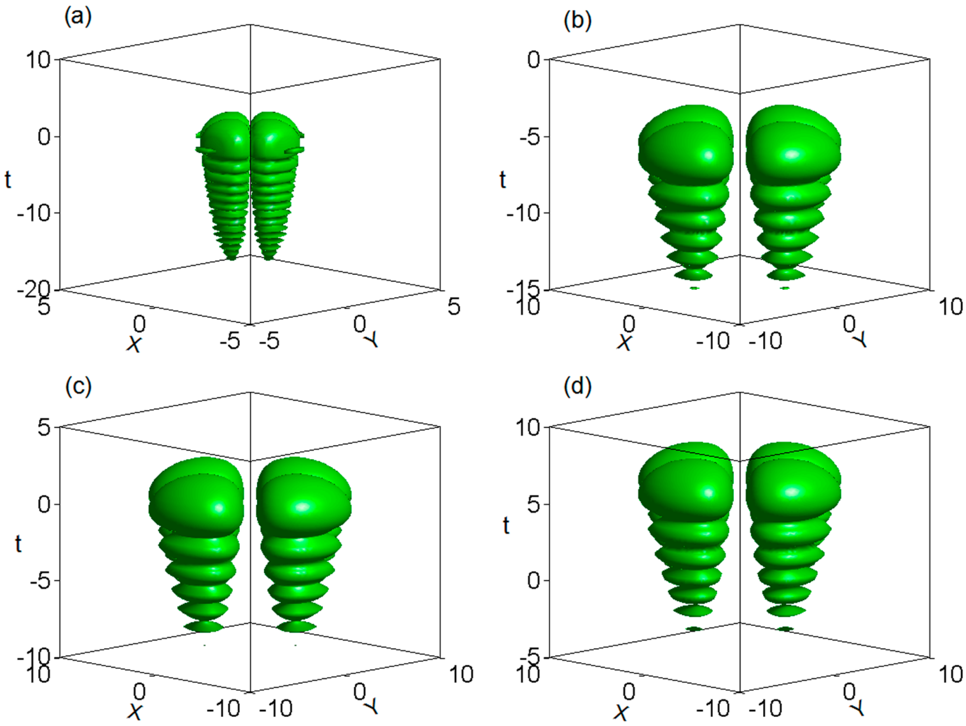

Figure 3. The 3D spatiotemporal even CAiBG wave packets with $m = 2$, taken (a) at $Z = 0$, (b)–(d) at $Z = 2$. (b)–(d) Represent $v_0 = -3$, $v_0 = 0$ and $v_0 = 3$, respectively.

We can see the initial CAiBG wave packets with several vortex rings in figure 2(a), which reduce to two vortex rings at $Z = 2$ in figures 2(b)–(d). By comparing to figures 2(b)–(d), we can obtain the decelerating vortex-Airy–Bessel–Gaussian wave packets in figure 2(b) and accelerating vortex-Airy–Bessel–Gaussian wave packets in figures 2(c) and (d). And different initial velocities hardly affect the structure in the plane domain during propagation.

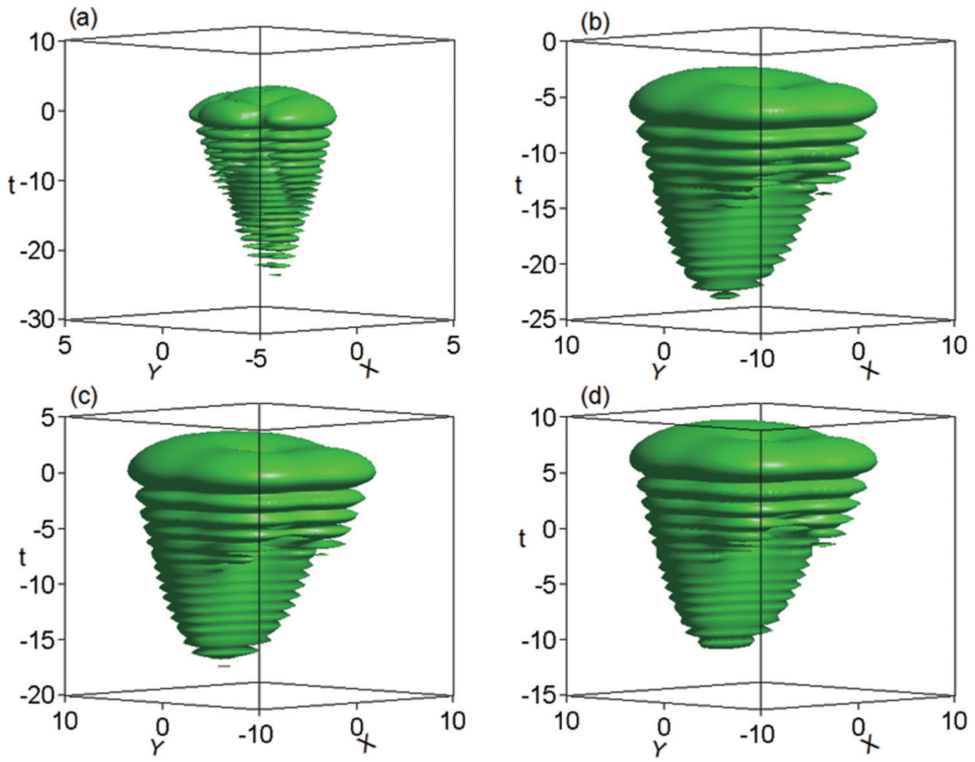

Figure 4. Iso-surface plots of the superposition of several CAiBG wave packets, taken (a) at $Z = 0$, (b)–(d) at $Z = 2$. (b)–(d) Represent $v_0 = -3$, $v_0 = 0$ and $v_0 = 3$, respectively.

3.2. Even and odd CAiBG wave packets

Besides the solution of vortex CAiBG wave packets, the even and odd CAiBG wave packets can be obtained as:

$$\begin{aligned} U^e &= \frac{1}{2}[U(m) + U(-m)] \\ &= c_0 e^{-\frac{2ih^2\beta_1}{Z-i}} J_m\left(\sqrt{\beta_1}\frac{-2i}{Z-i}hR\right)\frac{-i}{Z-i} e^{\frac{iR^2}{2(Z-i)}} \\ &\quad \times Ai\left(t + iaZ - v_0 Z - \frac{Z^2}{4}\right) e^{\left[a\left(t - v_0 Z - \frac{Z^2}{2}\right) - \frac{iZ^3}{12}\right]} \\ &\quad \times e^{\frac{i}{2}(a^2 Z - v_0^2 Z + 2v_0 t - v_0 Z^2 + tZ)} \cos(m\varphi). \end{aligned} \tag{14}$$

$$\begin{aligned} U^o &= \frac{1}{2i}[U(m) - U(-m)] \\ &= c_0 e^{-\frac{2ih^2\beta_1}{Z-i}} J_m\left(\sqrt{\beta_1}\frac{-2i}{Z-i}hR\right)\frac{-i}{Z-i} e^{\frac{iR^2}{2(Z-i)}} \\ &\quad \times Ai\left(t + iaZ - v_0 Z - \frac{Z^2}{4}\right) e^{\left[a\left(t - v_0 Z - \frac{Z^2}{2}\right) - \frac{iZ^3}{12}\right]} \\ &\quad \times e^{\frac{i}{2}(a^2 Z - v_0^2 Z + 2v_0 t - v_0 Z^2 + tZ)} \sin(m\varphi). \end{aligned} \tag{15}$$

The spatiotemporal even CAiBG wave packets are shown in figure 3. The 3D decelerating (figure 3(b)) and accelerating (figures 3(c) and (d)) necklace wave packets are obtained. We can find that each lobe contains $2m$ beads along the t axis. And the necklace wave packets are symmetric when m is an integer, which is asymmetric when m is a fraction.

3.3. The superposition of several CAiBG wave packets

According to the superpositions of the beams [30–32], we also consider what will happen to the superposition of several CAiBG wave packets:

$$\begin{aligned} U^s &= \sum_{m=1}^{4} U(m) \\ &= \sum_{m=1}^{4} c_0 e^{-\frac{2ih^2\beta_1}{Z-i}} J_m\left(\sqrt{\beta_1}\frac{-2i}{Z-i}hR\right)\frac{-i}{Z-i} e^{\frac{iR^2}{2(Z-i)}} \\ &\quad \times Ai\left(t + iaZ - v_0 Z - \frac{Z^2}{4}\right) e^{\left[a\left(t - v_0 Z - \frac{Z^2}{2}\right) - \frac{iZ^3}{12}\right]} \\ &\quad \times e^{\frac{i}{2}(a^2 Z - v_0^2 Z + 2v_0 t - v_0 Z^2 + tZ)} e^{im\varphi}. \end{aligned} \tag{16}$$

Figure 4 depicts the iso-surface plots of the intensity distributions of the superposition of several CAiBG wave packets. In figure 4(a), we can easily see the noncentrosymmetric wave packets in the initial plane. We can also find that the wave packets rotate during propagation in figures 4(b)–(d), which decelerate in figure 4(b) and accelerate in figures 4(c) and (d).

4. Conclusions

In summary, by solving the (3 + 1)D free-space Schrödinger equation in polar coordinates analytically, we have investigated the propagation of 3D CAiBG wave packets, even

CAiBG wave packets, odd CAiBG wave packets and the superposition of several CAiBG wave packets in free space. The CAiBG wave packets are constructed with the Airy pulses with initial velocity in temporal domain and the Bessel–Gaussian beams in space domain. Due to the initial velocity on Airy pulses, we can obtain decelerating and accelerating Airy–Bessel–Gaussian wave packets by selecting different initial velocities. Moreover, by superposing several CAiBG wave packets, we can obtain the rotating wave packets.

Our result is also applicable to generate other controllably decelerating and accelerating Airy wave packets with other beams in space domain in polar or Cartesian coordinates. And our demonstration may lead to potential applications of such wave packets in optical communications.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (11374108, 10904041, 11374107), and the Foundation of Cultivating Outstanding Young Scholars (Thousand, Hundred, Ten Program) of Guangdong Province in China. CAS Key Laboratory of Geospace Environment, University of Science and Technology of China, and Special Funds for the Cultivation of Guangdong College Students' Scientific and Technological Innovation (pdjh2016b0128).

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